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VOID NUCLEATION AND GROWTH FOR COMPRESSIBLE NON-LINEARLY ELASTIC MATERIALS: AN EXAMPLE

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Abstract—In this paper, we carry out an explicit analysis of a bifurcation problem for a solid sphere, composed of a special class of *compressible* non-linearly elastic materials, and subjected to prescribed radial stretch $\lambda > 1$ at its boundary. One solution to this problem, for all values of λ, is that of pure homogeneous stretching in which the sphere expands radially. However, for sufficiently large values of λ, a second configuration is possible where an internal traction-free spherical cavity forms at the origin. The critical stretch $\lambda = \lambda_{cr}$ at which this solution bifurcates from the trivial homogeneous solution is determined. The trivial solution is shown to become unstable at $\lambda = \lambda_{cr}$. It is also shown how the bifurcation model may be interpreted as describing sudden rapid growth of a *pre-existing* microvoid. The analogous issues for axisymmetric plane strain deformations of a cylinder are briefly discussed.

1. INTRODUCTION

Void nucleation and growth in solids have long been of concern because of the fundamental role such phenomena play in fracture and other failure mechanisms. [See e.g. Tvergaard (1990) for a recent review of void growth in metals.] Sudden void formation ("cavitation") in vulcanized rubber has also been observed experimentally by Gent and Lindley (1958). [See also Williams and Schapery (1965).] A recent review on cavitation in rubber is that of Gent (1990). Non-linear theories of solid mechanics have been extensively used recently to model such phenomena. The impetus for much of the current theoretical developments has been supplied by the work of Ball (1982). Ball has studied a class of *bifurcation problems* for the equations of non-linear elasticity which model the appearance of a cavity in the interior of an apparently solid homogeneous isotropic elastic sphere once a critical external load is attained. An alternative interpretation for such problems in terms of the sudden rapid growth of a *pre-existing* microvoid has been given by Horgan and Abeyaratne (1986); see also Sivaloganathan (1986a). As pointed out, for example, by Horgan and Abeyaratne (1986), cavitation is an inherently *non-linear* phenomenon and cannot be modeled using linearized solid mechanics theories.

In the comprehensive work of Ball (1982) on radially symmetric solutions, bifurcation and stability analyses are carried out for displacement and traction boundary-value problems in n-dimensions for both incompressible and compressible materials. For incompressible materials, the results are extensive and explicit while those in the more difficult compressible case are comparatively limited and require several constitutive assumptions. Further studies in the *compressible* case were carried out by Stuart (1985), Podio-Guidugli *et al.* (1986), Horgan and Abeyaratne (1986), Sivaloganathan (1986a, b), Ertan (1988) and Meynard (1990). Anisotropic compressible materials were considered by Antman and Negrón-Marrero (1987). Other contexts in which cavitation for compressible materials was investigated include consideration of non-radially symmetric solutions (James and Spector, 1989), elastodynamics (Pericak-Spector and Spector, 1988) and elastic membrane theory (Haughton, 1990; Steigmann, 1991; see also Haughton, 1986 for incompressible membrane theory). For *incompressible* materials, finite strain plasticity models were investigated by Chung *et al.* (1987) while the effects of rate dependence were examined by Abeyaratne and Hou (1989). Further studies for incompressible materials were carried out by Chou-Wang and Horgan (1989a) for elastostatics and by Chou-Wang and Horgan (1989b) for elastodynamics. The effects of material inhomogeneity on cavitation were investigated by Horgan and Pence (1989a, b, c). Void collapse for both incompressible and compressible materials has been examined by Abeyaratne and Hou (1991). Further work in plasticity was carried out by Hou and Abeyaratne (1991), Huang *et al.* (1991) and Tvergaard *et al.* (1990).

In contrast to the situation for incompressible materials, it is not possible, in general, to determine analytically solutions describing cavitation for compressible materials. Thus the analyses of cavitation for compressible materials have depended heavily on qualitative arguments for the relevant differential equations. Such arguments, in turn, have required imposition of several constitutive hypotheses [see e.g. Ball (1982), Stuart (1985), Podio-Guidugli *et al.* (1986), Sivaloganathan (1986a, b) and Meynard (1990)]. For the particular case of a Blatz-Ko material (which provides a model for a certain foamed rubber), explicit analytic solutions describing cavitation *have* been obtained by Horgan and Abeyaratne (1986) [see also Ertan (1988) for extensions of this work]. The purpose of the present paper is to examine another particular class of compressible materials for which the cavitated solutions can be obtained explicitly. The materials modeled by the strain-energy densities of concern here will be called *generalized Varga materials* [see e.g. Haughton (1987) for a special case] since they may be viewed as a generalization, to include the effect of compressibility, of an incompressible material model proposed by Varga (1966). It turns out that the solutions discussed here are simpler to describe than the corresponding solutions for the Blatz-Ko material (Horgan and Abeyaratne, 1986) and thus offer a particularly illuminating example of cavitation for isotropic compressible non-linearly elastic materials.

In the next section, we formulate the basic boundary-value problem that arises when a solid sphere, composed of a homogeneous isotropic compressible elastic solid, is subjected to a prescribed radial stretch $\lambda > 1$ on its boundary. One solution to this problem, *for all values of* λ, corresponds to a trivial homogeneous state in which the sphere expands radially. However, *for certain materials*, one can find, for sufficiently large values of λ, another possible radially symmetric configuration involving an internal traction-free spherical cavity. A class of materials (generalized Varga materials) for which this is possible is described in Section 3. A critical stretch λ_{cr} is determined such that when $\lambda > \lambda_{cr}$, such an additional solution is obtained. This solution bifurcates from the trivial solution at $\lambda = \lambda_{cr}$. (See the solid curve in Fig. 1.) To interpret physically the mathematical bifurcation problem described in Sections 2 and 3, we turn attention, in Section 4, to the problem of a *hollow* sphere with undeformed radii B, A ($B < A$). The inner surface is free of traction while the outer surface is subjected to a prescribed surface displacement, with $\lambda > 1$ denoting the prescribed stretch. The body is composed, again, of a generalized Varga material. The solution to this second problem is also obtained. Attention is then focussed on features of this solution in the limit as $B \to 0+$ corresponding to an infinitesimal microvoid. It is shown

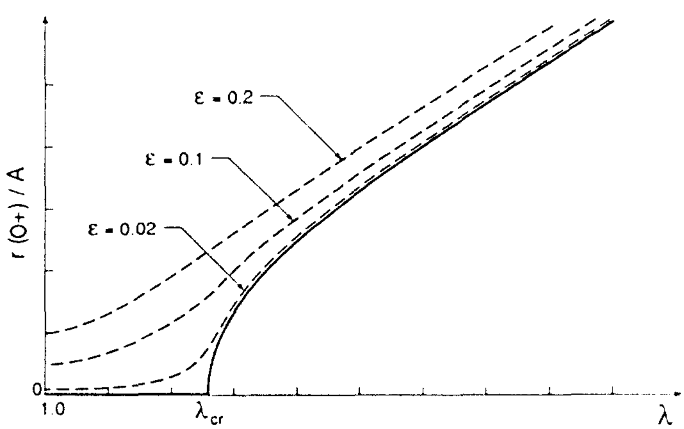

Fig. 1. Schematic diagram showing variation of the deformed cavity radius $r(0+)$ with prescribed stretch λ. The solid curve pertains to the bifurcation of a solid sphere (Sections 2, 3) while the dashed curves describe the growth of a pre-existing void for different undeformed void radii B (Section 4), with $B/A = \varepsilon$, $\varepsilon \ll 1$. The solid curve also depicts the limiting case of a microvoid ($B \to 0+$).

that, in this limit, the radius of the deformed cavity tends to zero for all values of the applied stretch in the range $1 < \lambda < \lambda_{cr}$, while for $\lambda > \lambda_{cr}$, this radius is positive and increases rapidly with increasing λ (see Fig. 1). Thus, in this limit, the solution for the hollow sphere coincides with that of the bifurcation problem. Consequently, as was shown, for example, by Horgan and Abeyaratne (1986) and Sivaloganathan (1986a) in related contexts, the bifurcation problem may be viewed as providing an idealized model describing the growth of a pre-existing microvoid. In Section 5, we examine the stability of the cavitated solution obtained in Section 3 using an energy comparison argument. It is shown that the potential energy associated with the cavitated solution, whenever it exists, is strictly less than that of the trivial solution corresponding to the same value of applied stretch and is also strictly less than that associated with any radial (not necessarily equilibrium) deformation. Thus, when $\lambda > \lambda_{cr}$, one indeed expects cavitation to occur. Finally, in Section 6 we briefly outline the corresponding results for plane strain deformations of a cylinder.

It should be noted that the constitutive assumptions made in Section 3 for the generalized Varga materials (12) are much less restrictive than those required in the analyses of Ball (1982), Sivaloganathan (1986a) or Podio-Guidugli *et al.* (1986). Thus the results obtained in these references using the direct method of the calculus of variations are not directly applicable to the bifurcation problem considered here. A similar remark pertains to the results obtained in Stuart (1985) and Meynard (1990) using shooting methods.

2. BIFURCATION PROBLEM FOR A SPHERE

We are concerned with a sphere composed of a homogeneous isotropic compressible non-linearly elastic material. Using spherical polar coordinates, the undeformed sphere occupies the domain $D_0 = \{(R, \Theta, \Phi) | 0 \leqslant R < A, 0 < \Theta \leqslant 2\pi, 0 \leqslant \Phi \leqslant \pi\}$. The sphere is subjected to a prescribed uniform radial displacement at its surface $R = A$. The resulting deformation is a mapping which takes the point with spherical polar coordinates (R, Θ, Φ) in the undeformed region D_0 to the point (r, θ, ϕ) in the deformed region D. We assume that the deformation is radially symmetric so that $\theta = \Theta$, $\phi = \Phi$ and

$$r = r(R) > 0, \quad 0 < R \leqslant A; \quad r(0+) \geqslant 0, \tag{1}$$

where $r = r(R)$ is to be determined. If $r(0+) = 0$, the sphere remains solid. However, if $r(0+) > 0$, then a spherical cavity of radius $r(0+)$ has formed at the origin. In this event, the cavity surface is assumed to be traction-free. The polar components of the deformation gradient tensor $\mathbf{F}$ associated with the deformation at hand are given by

$$\mathbf{F} = \text{diag}\,(\dot{r}(R), r(R)/R, r(R)/R), \tag{2}$$

while the principal stretches are

$$\lambda_1 = \dot{r}(R), \quad \lambda_2 = \lambda_3 = r(R)/R. \tag{3}$$

In (2) and (3) the superposed dot denotes differentiation with respect to R. It is assumed that $J \equiv \det \mathbf{F} = r^2\dot{r}/R^2 > 0$ on $0 < R \leqslant A$, so that, in view of (1), we have

$$\dot{r}(R) > 0, \quad 0 < R \leqslant A. \tag{4}$$

The strain-energy density per unit undeformed volume for isotropic compressible elastic materials is denoted by $W = W(\lambda_1, \lambda_2, \lambda_3)$, where λ_i $(i = 1, 2, 3)$ are the principal stretches. The function W is invariant with respect to interchange of the λ_i and is normalized so that it vanishes in the undeformed state and so $W(1, 1, 1) = 0$. The principal components of the Cauchy stress $\mathbf{T}$ are given by

$$T_{ii} = \frac{\lambda_i}{\lambda_1 \lambda_2 \lambda_3} \frac{\partial W}{\partial \lambda_i} \text{ (no sum on } i\text{)}. \tag{5}$$

In view of (3), we thus have

$$T_{rr} = \frac{1}{\lambda_2 \lambda_3} \frac{\partial W}{\partial \lambda_1}, \quad T_{\theta\theta} = T_{\phi\phi} = \frac{1}{\lambda_1 \lambda_3} \frac{\partial W}{\partial \lambda_2}, \tag{6}$$

for the radial and hoop stresses. Note that in (6) we consider $\mathbf{T}(R)$ rather than the more conventional $\mathbf{T}(r)$.

In the absence of body forces, the equilibrium equations $\operatorname{div} \mathbf{T} = 0$ can be shown to reduce to the single equation

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2 \frac{\partial W}{\partial \lambda_1}\right) - 2R\frac{\partial W}{\partial \lambda_2} = 0, \quad 0 < R < A, \tag{7}$$

where W is evaluated at the principal stretches (3). By virtue of (3), we see that (7) is a second-order non-linear ordinary differential equation for $r(R)$. Since the sphere is subjected to prescribed uniform radial displacement of its boundary $r = A$, we have the boundary condition

$$r(A) = \lambda A, \tag{8}$$

where $\lambda > 1$ is the prescribed radial stretch. At the origin, we have *either*

$$r(0+) = 0 \tag{9}$$

or

$$T_{rr}(0+) = 0 \quad \text{if } r(0+) > 0. \tag{10}$$

It may be readily verified that one solution to the problem (7)–(10), for *all* values of $\lambda > 1$ is

$$r(R) = \lambda R, \quad 0 \leqslant R \leqslant A. \tag{11}$$

Note that (9) is satisfied in this case. This homogeneous solution, which we call the *trivial solution*, corresponds to a homogeneous deformation in which the sphere expands radially. For *certain materials* and for λ *sufficiently large*, it has been shown that, in addition to the trivial solution, there exists a second solution for which (10) holds so that a traction-free cavity has formed at the origin. The existence of such solutions describing *cavitation* has been established for a wide class of isotropic compressible materials by Ball (1982), Stuart (1985), Podio-Guidugli *et al.* (1986), Horgan and Abeyaratne (1986), Sivaloganathan (1986a, b) and Meynard (1990) and for *anisotropic* compressible materials by Antman and Negrón-Marrero (1987). The cavitation solutions have been shown to bifurcate from the trivial solution at the critical value λ_{cr} at which the trivial solution becomes unstable.

It is not possible, in general, to determine the cavitation solutions analytically. For the particular case of a Blatz-Ko material, analytic solutions describing cavitation *have* been obtained by Horgan and Abeyaratne (1988) [see also Etan (1988) for extensions of this work]. Horgan and Abeyaratne (1986) treat the analogous plane strain problem for cylindrical cavities in detail, while the spherical problem can be analyzed using the spherically symmetric solutions developed in Chung *et al.* (1986) [see note added in proof in Horgan and Abeyaratne (1986)]. The spherical problem has been subsequently re-investigated by Tian-hu (1990) confirming the results stated in Horgan and Abeyaratne

(1986). The purpose of the present paper is to examine another particular class of compressible materials for which the cavitated solutions can be obtained explicitly. It turns out that the solutions are simpler to describe than the corresponding results for the Blatz-Ko material, and thus offer a particularly illuminating example of cavitation for isotropic compressible non-linearly elastic materials.

3. SOLUTION FOR A PARTICULAR CLASS OF MATERIALS

Consider the class of isotropic compressible materials for which W has the form

$$W = c_1(\lambda_1+\lambda_2+\lambda_3-3)+c_2(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1-3)+g(\lambda_1\lambda_2\lambda_3), \tag{12}$$

where c_1, c_2 are constants and g is an arbitrary sufficiently smooth function of its argument. We assume that g satisfies the normalization conditions

$$g(1) = 0, \quad g'(1) = -(2c_2+c_1), \tag{13}$$

so that W and the Cauchy stress $\mathbf{T}$ vanish in the undeformed state. Materials of the form (12) have been extensively analyzed in Carroll (1988), where closed form analytical representations for a wide variety of deformations have been obtained. When $c_2 = 0$ in (12), the materials modeled by (12) have been called Varga materials by Haughton (1987) since they may be viewed as a generalization, to include the effect of compressibility, of an incompressible material model proposed by Varga (1966). For $c_2 \neq 0$, we call the materials (12) *generalized Varga materials*.

When W is given by (12), the differential equation (7) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}R}g'(J) = 0, \quad 0 < R < A, \tag{14}$$

where

$$J = I_3^{1/2} = \lambda_1\lambda_2\lambda_3 = r^2\dot{r}/R^2. \tag{15}$$

Using the chain rule, we may write (14) as

$$g''(J)\frac{\mathrm{d}}{\mathrm{d}R}J = 0, \quad 0 < R < A, \tag{16}$$

and so, provided

$$g''(J) \neq 0, \quad 0 < R < A, \tag{17}$$

we deduce from (16) that

$$J = r^2\dot{r}/R^2 = k_1 \quad \text{on } 0 < R < A, \tag{18}$$

where k_1 is a constant. By virtue of (4), we have $k_1 > 0$. Equation (18) may be integrated to yield

$$r^3(R) = k_1R^3+k_2, \tag{19}$$

where k_2 is a constant. Thus (19) is an explicit closed form solution to (7) for the material (12), provided (17) holds. The solution (19) was obtained by Carroll (1988) and independently by Horgan (1989). It was also established in Haughton (1987) for the case when $c_2 = 0$.

We turn now to the boundary conditions. First we note that if $k_2 = 0$, $k_1 = \lambda^3$ in (19), we recover the trivial solution (11) for which (8) and (9) hold. Suppose now that (8) and (10) are to be satisfied. Then k_1, k_2 must be such that

$$\lambda^3 A^3 = k_1 A^3 + k_2, \tag{20}$$

$$k_2 > 0, \tag{21}$$

the latter condition being equivalent to $r(0+) > 0$. The radial stress T_{rr} follows from (6), (3) and (12) as

$$T_{rr} = \frac{R^2}{r^2(R)} c_1 + 2\frac{R}{r(R)} c_2 + g'(J), \tag{22}$$

where $J = r^2 \dot{r}/R^2$. Since $J = k_1$ from (18) and since $r(0+) > 0$, we see from (22) that (10) is satisfied provided

$$g'(k_1) = 0. \tag{23}$$

Thus, if a positive constant k_1 can be found such that (23) *holds, the radial stress corresponding to* (19) *vanishes at the cavity boundary.* In view of (18) and (23) we then have $g'(J) = 0$ and so the radial stress field (22) is

$$T_{rr} = \frac{R^2}{r^2(R)} c_1 + \frac{2R}{r(R)} c_2, \tag{24}$$

where $r(R)$ is given by (19). From (20) we have

$$k_2 = (\lambda^3 - k_1) A^3, \tag{25}$$

and so (21) will hold if and only if

$$\lambda > (k_1)^{1/3}. \tag{26}$$

Thus the *critical stretch* λ_{cr} is given by

$$\lambda_{cr} = (k_1)^{1/3}, \tag{27}$$

where $k_1 > 0$ satisfies (23). To ensure the existence of λ_{cr}, we thus assume henceforth that the constitutive function g in (12) is such that (23) holds for some unique positive constant k_1. In fact, since the applied stretch $\lambda > 1$, we require that $k_1 > 1$. A *sufficient* condition to ensure the existence of such a k_1 is to assume that $g(s)$ has a *single minimum* at $s = k_1 (k_1 > 1)$. Henceforth we assume that this is the case. Further assumptions on g are usually necessary to ensure a physically reasonable response but we shall not pursue this here.

In summary, *for the material* (12) *with g satisfying* (13) *and* (17) *and the stipulations of the preceding paragraph, we have found that whenever the prescribed stretch λ is greater than λ_{cr}, the existence of a solution with internal traction-free cavity is guaranteed and this solution is given by*

$$r^3(R) = k_1(R^3 - A^3) + \lambda^3 A^3, \tag{28}$$

where $k_1 > 0$ is determined from (23).

The deformed cavity radius $r(0+)$ follows from (28) as

$$r(0+)/A = (\lambda^3 - k_1)^{1/3}, \quad \lambda \geq \lambda_{cr}. \tag{29}$$

The variation of the cavity radius with prescribed stretch λ, as described by (29), is depicted by the solid curve in Fig. 1 (in the range $\lambda > \lambda_{cr}$).

The radial stress field subsequent to cavitation is given by (24) with $r(R)$ given by (28). The critical value T_{rr}^{cr} of the radial stress at the outer boundary when $\lambda = \lambda_{cr}$ follows from (24), (27) and (28) as

$$T_{rr}^{cr} = c_1 k_1^{-2/3} + 2c_2 k_1^{-1/3}. \tag{30}$$

The hoop stresses subsequent to cavitation follow from (6), (3), (12) and (18) as

$$T_h \equiv T_{\theta\theta} = T_{\phi\phi} = \frac{c_1 r(R)}{k_1 R} + c_2 \left[\frac{R}{r(R)} + \frac{r^2(R)}{k_1 R^2}\right], \tag{31}$$

where $r(R)$ is given by (28). The critical value T_h^{cr} of the hoop stress at the outer boundary when $\lambda = \lambda_{cr}$ then follows from (31), (27) and (28) as

$$T_h^{cr} = c_1 k_1^{-2/3} + 2c_2 k_1^{-1/3}. \tag{32}$$

We observe that this value coincides with (30).

We remark that the equilibrium solutions described above could also have been obtained from an energy minimization approach. Since displacements are prescribed at the outer surface of the sphere and the cavity surface (when it exists) is free of traction, the associated potential energy E is given by

$$E = \int_V W \, dV = 4\pi \int_0^A R^2 W \, dR$$
$$\equiv 4\pi \int_0^A F(r, \dot{r}; R) \, dR. \tag{33}$$

Minimizers of E necessarily satisfy the Legendre inequality

$$\frac{\partial^2 F}{\partial (\dot{r})^2} > 0 \text{ for all } R \text{ in } 0 < R < A. \tag{34}$$

When the form of W is recalled from (12), it is seen that (34) reduces to

$$g''(J) > 0, \tag{35}$$

so that $g(\cdot)$ must be convex at the value J associated with the minimizer. If one adopted as a constitutive hypothesis that (35) should hold for *all* $J > 0$, then this would guarantee the existence of a single *minimum* for $g(s)$ at $s = k_1$ ($k_1 > 1$), as assumed above.

4. GROWTH OF A MICROVOID

It has been shown by Horgan and Abeyaratne (1986) (in the context of plane deformations of a Blatz-Ko material), and by Sivaloganathan (1986a) (for radially symmetric deformations of classes of compressible materials) that bifurcation problems of the type just discussed may be viewed as providing an idealized model describing the growth of a pre-existing microvoid. The latter problem concerns a *hollow* sphere with undeformed inner and outer radii B, A, respectively. The outer boundary is subjected to a prescribed uniform radial displacement as before, while the inner boundary is traction-free. Thus, the differential equation (7) now holds on the interval $B < R < A$ and for the material (12) we have the solution

$$r^3(R) = K_1R^3 + K_2, \quad \text{on } B < R < A, \tag{36}$$

where K_1 and K_2 are constants. The boundary conditions are

$$r(A) = \lambda A, \tag{37}$$

and

$$\frac{B^2c_1}{r^2(B)} + \frac{2Bc_2}{r(B)} + g'(K_1) = 0, \tag{38}$$

where (22), (18) have been used to obtain (38). The two constants K_1, K_2 are thus determined from

$$\lambda^3A^3 = K_1A^3 + K_2, \tag{39}$$

and

$$\frac{B^2c_1}{(K_1B^3+K_2)^{2/3}} + \frac{2Bc_2}{(K_1B^3+K_2)^{1/3}} + g'(K_1) = 0. \tag{40}$$

On solving (39) for K_2 and substituting in (40) we obtain

$$\frac{B^2c_1}{[(\lambda^3-K_1)A^3+K_1B^3]^{2/3}} + \frac{2Bc_2}{[(\lambda^3-K_1)A^3+K_1B^3]^{1/3}} + g'(K_1) = 0. \tag{41}$$

Using the dimensionless radius ratio

$$\alpha = B/A, \quad 0 < \alpha < 1, \tag{42}$$

we write (41) as

$$\frac{c_1}{[(\lambda^3-K_1)\alpha^{-3}+K_1]^{2/3}} + \frac{2c_2}{[(\lambda^3-K_1)\alpha^{-3}+K_1]^{1/3}} + g'(K_1) = 0. \tag{43}$$

For prescribed λ, with c_1, c_2 and α known, (43) is an equation for the determination of K_1. Once K_1 is found, K_2 may be obtained from (39) and thus the deformation given by (36) is determined.

The deformed inner radius is given by

$$r^3(B) = K_1B^3 + K_2 = K_1B^3 + (\lambda^3 - K_1)A^3, \tag{44}$$

and so

$$\frac{r(B)}{A} = [K_1\alpha^3 + (\lambda^3 - K_1)]^{1/3}, \tag{45}$$

where K_1 (which depends, for given c_1, c_2, on λ, α) is determined from (43). The $r(B)/A$ versus λ relationship is thus implicitly given by (45), together with (43).

We turn now to the examination of the limiting case of a microvoid. Thus we consider the preceding solution in the limit as $B \to 0+$, with A, λ, R held fixed. By (42) we see that $\alpha \to 0+$ as $B \to 0+$. Letting $\alpha \to 0+$ in (45), we see that, when $\lambda^3 > K_1$, (45) and (43) reduce to (29) and (23) respectively (with $K_1 \equiv k_1$). On the other hand, when $\lambda^3 \leqslant K_1$, on letting $\alpha \to 0+$ in (45) we see that K_1 must be equal to λ^3 since $r(B)$ must be non-negative. Thus $\lambda^3 = K_1$ and so from (39) we see that $K_2 = 0$. On setting $K_1 = \lambda^3$ and $K_2 = 0$ in (36), one recovers the *trivial solution* (11). Thus, we see that in the limit as $B \to 0+$, the solution for the microvoid problem tends to the trivial solution as long as $\lambda \leqslant \lambda_{cr}$, while it tends to the cavitated solution when $\lambda > \lambda_{cr}$. The foregoing limit is depicted schematically in Fig. 1 where the dashed curves describe growth of a pre-existing void for different undeformed void radii B. The solid curve denotes the limiting case of a microvoid ($B \to 0+$). Similar considerations have been discussed by Horgan and Abeyaratne (1986) and Sivaloganathan (1986a).

5. ENERGY COMPARISON

We now return to the solutions described in Section 3. Since for values of $\lambda > \lambda_{cr}$ we have obtained two solutions, it is natural to compare their associated *energies* at the same value of prescribed stretch. [Cf. Ball (1982).] Since displacements are prescribed at the outer surface of the sphere and the cavity surface (when it exists) is free of traction, the associated potential energy E is given by

$$E = \int_V W \, dV = 4\pi \int_0^A R^2 W \, dR. \tag{46}$$

It is readily verified from (3) that

$$\frac{d\lambda_2}{dR} = \frac{1}{R}(\lambda_1 - \lambda_2). \tag{47}$$

Making use of (47), it can be shown that (7) may be rewritten as

$$\frac{d}{dR}\left\{R^3\left[W - (\lambda_1 - \lambda_2)\frac{\partial W}{\partial \lambda_1}\right]\right\} - 3R^2 W = 0. \tag{48}$$

[Cf. Ball (1982), eqn (6.12). As remarked there, (48) also follows from a conservation law due to Green (1973).] Thus from (48) we find that

$$4\pi \int_0^A R^2 W \, dR = \frac{4\pi}{3}\left[R^3\left\{W - (\lambda_1 - \lambda_2)\frac{\partial W}{\partial \lambda_1}\right\}\right]_{R=A}, \tag{49}$$

and so

$$E = \frac{4\pi}{3} A^3 \left\{W(\dot{r}(A), \lambda, \lambda) - [\dot{r}(A) - \lambda]\frac{\partial W}{\partial \lambda_1}(\dot{r}(A), \lambda, \lambda)\right\}, \tag{50}$$

where $\lambda = r(A)/A$ is the prescribed stretch at the outer boundary. For the trivial solution (11), we thus obtain $E = E_t$ with

$$E_t = \frac{4\pi}{3} A^3 W(\lambda, \lambda, \lambda). \tag{51}$$

For the cavitated solution, it follows from (8) and (19) that $\dot{r}(A) = k_1/\lambda^2$ and so $E = E_c$ with

$$E_c = \frac{4\pi}{3} A^3 \left[W\left(\frac{k_1}{\lambda^2}, \lambda, \lambda\right) - \left(\frac{k_1}{\lambda^2} - \lambda\right) \frac{\partial W}{\partial \lambda_1}\left(\frac{k_1}{\lambda^2}, \lambda, \lambda\right) \right]. \tag{52}$$

When the explicit form (12) for the strain-energy density is used in (51) and (52), one finds that

$$E_c - E_t = \frac{4\pi A^3}{3}[g(k_1) - g(\lambda^3)]. \tag{53}$$

Thus for $\lambda > \lambda_{cr}$ (i.e. $\lambda^3 > k_1$), we deduce from (53) that

$$E_c < E_t, \tag{54}$$

since we have assumed (recall Section 3) that $g(s)$ has a minimum at $s = k_1$. Thus, the energy associated with the cavitated solution (whenever it exists) is strictly less than that of the trivial solution corresponding to the same value of λ.

In fact, a much stronger result can be established, namely that the cavitated equilibrium solution minimizes the energy absolutely relative to *any* radial (not necessarily equilibrium) deformation. To see this we write

$$W = W(\dot{r}, \lambda, \lambda) = c_1(\dot{r} + 2r/R - 3) + c_2[2r\dot{r}/R + (r/R)^2 - 3] + g(J). \tag{55}$$

The term involving c_1 is $c_1 R^{-2}(R^2 r)^{\boldsymbol{\cdot}}$ and the term involving c_2 is $2c_2 r R^{-1/2}(R^{1/2} r)^{\boldsymbol{\cdot}}$, to within unimportant additive constant terms. When integrated to obtain the total energy, these terms integrate to $c_1 A^2 r(A)$ and $c_2 A r^2(A)$ respectively, and are fixed since $r(A)$ is prescribed as in (8). Consequently, for *any* radial deformation $r(R)$ with $r(A) > k_1^{1/3} A$, it follows that

$$(4\pi)^{-1}(E - E_c) = \int_0^A R^2[g(J) - g(k_1)]\, \mathrm{d}R. \tag{56}$$

Thus E_c is the absolute minimum since k_1 minimizes $g(\cdot)$. Therefore, *within the class of radial deformations*, the cavitated solution yields the absolute minimum energy. Thus when $\lambda > \lambda_{cr}$, conditions are indeed energetically favorable for a cavity to appear. (It would be of interest to consider energy minimization within the wider class of not necessarily radial or radially symmetric deformations but this is beyond the scope of the present work.)

6. RESULTS FOR A CYLINDER IN PLANE STRAIN

In this section, we describe briefly how the foregoing considerations can be applied to the analogous problem for a cylinder.

6.1. *Bifurcation problem*

Using cylindrical polar co-ordinates, the undeformed cross-section of the cylinder occupies the domain $D_0 = \{(R, \Theta) | 0 \leqslant R < A, 0 < \Theta \leqslant 2\pi\}$. The cylinder is subjected to a prescribed uniform radial displacement at its surface $R = A$. The deformation taking the point (R, Θ) to (r, θ) is assumed to be an axisymmetric plane strain so that $\theta = \Theta$ and

$$r = r(R) > 0, \quad 0 < R \leqslant A; \quad r(0+) \geqslant 0. \tag{57}$$

As before, if $r(0+) = 0$ the cylinder remains solid while if $r(0+) > 0$, a cavity (whose surface is assumed traction-free) has formed at the origin. We have

$$\mathbf{F} = \operatorname{diag}\{\dot{r}(R), r(R)/R\}, \tag{58}$$

with principal stretches

$$\lambda_1 = \dot{r}(R), \quad \lambda_2 = r(R)/R, \tag{59}$$

and assuming $J \equiv \det \mathbf{F} = r\dot{r}/R > 0$ on $0 < R \leqslant A$, we thus have

$$\dot{r}(R) > 0, \quad 0 < R \leqslant A. \tag{60}$$

Denoting the strain-energy density per unit undeformed volume for the isotropic compressible materials at hand by $W(\lambda_1, \lambda_2)$, with $W(1, 1) = 0$, we have

$$T_{\alpha\alpha} = \frac{\lambda_\alpha}{\lambda_1\lambda_2}\frac{\partial W}{\partial \lambda_\alpha} \text{ (no sum on } \alpha). \tag{61}$$

Thus

$$T_{rr} = \frac{1}{\lambda_2}\frac{\partial W}{\partial \lambda_1}, \quad T_{\theta\theta} = \frac{1}{\lambda_1}\frac{\partial W}{\partial \lambda_2}, \tag{62}$$

and the equilibrium equations reduce to

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\partial W}{\partial \lambda_1}\right) - \frac{\partial W}{\partial \lambda_2} = 0, \quad 0 < R < A, \tag{63}$$

where W is evaluated at the principal stretches (59). The boundary condition at $r = A$ is

$$r(A) = \lambda A, \tag{64}$$

where $\lambda > 1$ is the prescribed radial stretch. At the origin, we have *either*

$$r(0+) = 0 \tag{65}$$

or

$$T_{rr}(0+) = 0 \quad \text{if } r(0+) > 0. \tag{66}$$

As in the case of the sphere, it is again easily verified that one solution to the problem (63)–(66) for *all* values of λ is the trivial solution

$$r(R) = \lambda R, \quad 0 \leqslant R \leqslant A, \tag{67}$$

describing uniform radial expansion.

6.2. *Solution for a particular class of materials*

We again consider the strain-energy density (12) now specialized to plane deformations. The differential equation (63) in this case reduces to

$$\frac{\mathrm{d}}{\mathrm{d}R}g'(J) = 0, \quad 0 < R < A, \tag{68}$$

where

$$J = \lambda_1\lambda_2 = r\dot{r}/R. \tag{69}$$

As before, we deduce from (68) that, provided

$$g''(J) \neq 0, \quad 0 < R < A, \tag{70}$$

then

$$J = r\dot{r}/R = k_1 \quad \text{on } 0 < R < A, \tag{71}$$

where $k_1 > 0$ is a constant. This leads to the explicit solution

$$r^2(R) = k_1 R^2 + k_2, \tag{72}$$

where k_2 is a constant. The solution (72) was also obtained by Carroll (1988) and independently by Horgan (1989).

In what follows, we confine attention to the special Varga material for which $c_2 = 0$ in (12). Proceeding as in the three-dimensional case for the sphere, it is readily shown that *if a positive constant k_1 can be found such that*

$$g'(k_1) = 0, \tag{73}$$

then for

$$\lambda > \lambda_{cr}, \quad \lambda_{cr} \equiv (k_1)^{1/2}, \tag{74}$$

the existence of a solution with an internal traction-free cavity is guaranteed and this solution is given by

$$r^2(R) = k_1(R^2 - A^2) + \lambda^2 A^2. \tag{75}$$

The deformed cavity radius $r(0+)$ follows from (75) as

$$r(0+)/A = (\lambda^2 - k_1)^{1/2}, \quad \lambda \geq \lambda_{cr}. \tag{76}$$

The graph of $r(0+)/A$ versus λ, while now of quadratic nature rather than cubic as in (29), will have the general features shown in Fig. 1. Again, to ensure that a unique constant k_1 exists such that (73) holds, it is sufficient to assume that $g(s)$ has a single minimum at $s = k_1$ ($k_1 > 1$). The considerations of Sections 4 and 5 may be adapted in an obvious way to the plane strain problem—we omit the details.

It should be noted that in a recent paper on cavitation for compressible elastic *membranes* (Haughton, 1990), Haughton has numerically demonstrated that cavitation can occur in a membrane composed of a Blatz-Ko material. However, he also shows analytically that for the special Varga material ((12) with $c_2 = 0$) cavitation *cannot* occur. The membrane theory employed by Haughton (1990) is derived from three-dimensional elasticity theory. Subsequently, Steigmann (1991) has shown that cavitation *can* occur for this material if a *direct* membrane theory is employed. As noted by Steigmann (1991), his analysis also pertains to the plane strain problem.

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REFERENCES

Abeyaratne, R. and Hou, H.-S. (1989). Growth of an infinitesimal cavity in a rate-dependent solid. *J. Appl. Mech* **56**, 40–46.

Abeyaratne, R. and Hou, H.-S. (1991). Void collapse in an incompressible elastic solid (to be published).

Antman, S. S. and Negrón-Marrero, P. V. (1987). The remarkable nature of radially symmetric equilibrium states of aeolotropic nonlinearly elastic bodies. *J. Elasticity* **18**, 131–164.

Ball, J. M. (1982). Discontinuous equilibrium solutions and cavitation in nonlinear elasticity. *Phil. Trans. R. Soc. Lond.* **A306**, 557–610.

Carroll, M. M. (1988). Finite strain solutions in compressible isotropic elasticity. *J. Elasticity* **20**, 65–92.

Chou-Wang, M.-S. and Horgan, C. O. (1989a). Void nucleation and growth for a class of incompressible nonlinearly elastic materials. *Int. J. Solids Structures* **25**, 1239–1254.

Chou-Wang, M.-S. and Horgan, C. O. (1989b). Cavitation in nonlinear elastodynamics for neo-Hookean materials. *Int. J. Engng Sci.* **27**, 967–973.

Chung, D.-T., Horgan, C. O. and Abeyaratne, R. (1986). The finite deformation of internally pressurized hollow cylinders and spheres for a class of compressible elastic materials. *Int. J. Solids Structures* **22**, 1557–1570.

Chung, D.-T., Horgan, C. O. and Abeyaratne, R. (1987). A note on a bifurcation problem in finite plasticity related to void nucleation. *Int. J. Solids Structures* **23**, 983–988.

Ertan, N. (1988). Influence of compressibility and hardening on cavitation. *ASCE J. Engng Mech.* **114**, 1231–1244.

Gent, A. N. (1990). Cavitation in rubber: a cautionary tale. *Rubber Chem. Technol.* **63**, G49–G53.

Gent, A. N. and Lindley, P. B. (1958). Internal rupture of bonded rubber cylinders in tension. *Proc. R. Soc. Lond.* **A249**, 195–205.

Green, A. E. (1973). On some general formulae in finite elastostatics. *Arch. Ration. Mech. Anal.* **50**, 73–80.

Haughton, D. M. (1986). On non-existence of cavitation in incompressible elastic membranes. *Q. J. Mech. Appl. Math.* **39**, 289–296.

Haughton, D. M. (1987). Inflation of thick-walled compressible elastic spherical shells. *IMA J. Appl. Math.* **39**, 259–272.

Haughton, D. M. (1990). Cavitation in compressible elastic membranes. *Int. J. Engng Sci.* **28**, 163–168.

Horgan, C. O. (1989). Some remarks on axisymmetric solutions in finite elastostatics for compressible materials. *Proc. R. Irish Academy* **89A**, 185–193.

Horgan, C. O. and Abeyaratne, R. (1986). A bifurcation problem for a compressible nonlinearly elastic medium: growth of a micro-void. *J. Elasticity* **16**, 189–200.

Horgan, C. O. and Pence, T. J. (1989a). Void nucleation in tensile dead-loading of a composite incompressible nonlinearly elastic sphere. *J. Elasticity* **21**, 61–82.

Horgan, C. O. and Pence, T. J. (1989b). Cavity formation at the center of a composite incompressible nonlinearly elastic sphere. *J. Appl. Mech.* **56**, 302–308.

Horgan, C. O. and Pence, T. J. (1989c). Void nucleation due to large deformations in nonlinearly elastic composites. *Proceedings of the 4th Japan–US Conference on Composite Materials*, Washington, DC, June 1988 (edited by J. R. Vinson), pp. 232–241. Technomic Publishing Co., Lancaster, PA.

Hou, H.-S. and Abeyaratne, R. (1991). Cavitation in elastic and elastic-plastic solids. *J. Mech. Phys. Solids* (in press).

Huang, Y., Hutchinson, J. W. and Tvergaard, V. (1991). Cavitation instabilities in elastic-plastic solids. *J. Mech. Phys. Solids* **39**, 223–242.

James, R. D. and Spector, S. J. (1989). The formation of filamentary voids in solids. IMA Preprint Series, No. 572, University of Minnesota.

Meynard, F. (1990). Radially symmetric cavitation and non-cavitation of a hyperelastic ball. Preprint, EPFL-DMA, Lausanne, Switzerland.

Pericak-Spector, K. A. and Spector, S. J. (1988). Nonuniqueness for a hyperbolic system: cavitation in nonlinear elastodynamics. *Arch. Rat. Mech. Anal.* **101**, 293–317.

Podio-Guidugli, P., Vergara Caffarelli, G. and Virga, E. G. (1986). Discontinuous energy minimizers in nonlinear elastostatics: an example of J. Ball revisited. *J. Elasticity* **16**, 75–96.

Sivaloganathan, J. (1986a). Uniqueness of regular and singular equilibria for spherically symmetric problems of nonlinear elasticity. *Arch. Ration. Mech. Analysis* **96**, 97–136.

Sivaloganathan, J. (1986b). A field theory approach to stability of radial equilibria in nonlinear elasticity. *Math. Proc. Camb. Phil. Soc.* **99**, 589–604.

Steigmann, D. J. (1991). Cavitation in elastic membranes—an example. *J. Elasticity* (in press).

Stuart, C. A. (1985). Radially symmetric cavitation for hyperelastic materials. *Ann. Inst. Henri Poincaré-Analyse non lineaire* **2**, 33–66.

Tian-hu, H. (1990). A theory of the appearance and growth of the micro-spherical void. *Int. J. Fracture* **43**, R51–R55.

Tvergaard, V. (1990). Material failure by void growth to coalescence. In *Advances in Applied Mechanics* (edited by J. Hutchinson and T. Wu), Vol. 27, pp. 83–151. Academic Press, San Diego.

Tvergaard, V., Huang, Y. and Hutchinson, J. W. (1990). Cavitation instabilities in a power hardening elastic-plastic solid, Tech. Univ. of Denmark, Report #415.

Varga, O. H. (1966). *Stress–Strain Behavior of Elastic Materials.* Wiley, New York.

Williams, M. L. and Schapery, R. A. (1965). Spherical flaw instability in hydrostatic tension. *Int. J. Fracture Mechanics* **1**, 64–71.